Review and Summary

Design of Prismatic Beams

This chapter was devoted to the analysis and design of beams under transverse loadings consisting of concentrated or distributed loads. The beams are classified according to the way they are supported (Fig. 5.20). Only *statically determinate* beams were considered, where all support reactions can be determined by statics.

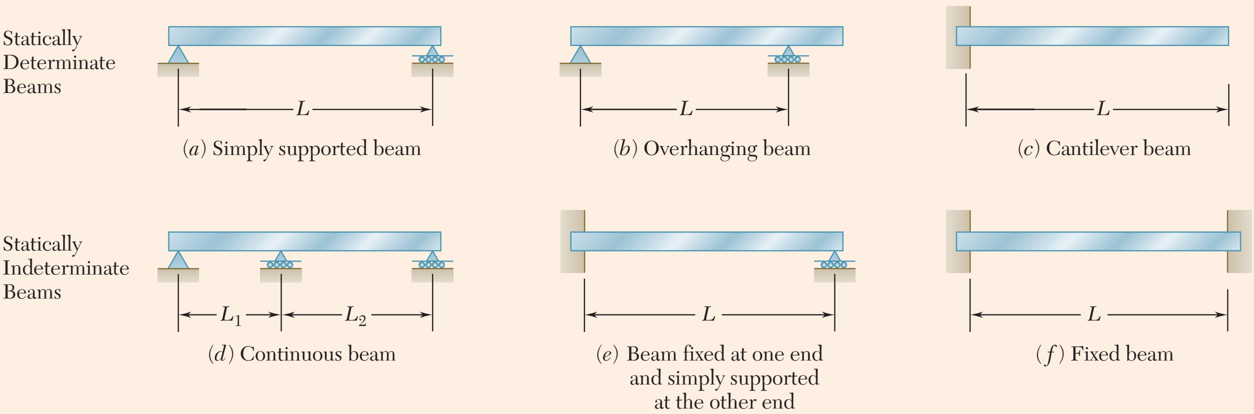


Fig. 5.20 Common beam support configurations.

Normal Stresses Due to Bending

While transverse loadings cause both bending and shear in a beam, the normal stresses caused by bending are the dominant criterion in the design of a beam for strength [Sec. 5.1]. Therefore, this chapter dealt only with the determination of the normal stresses in a beam, the effect of shearing stresses being examined in the next one.

The flexure formula for the determination of the maximum value σ_m of the normal stress in a given section of the beam is

$$\sigma_m = \frac{|M|c}{I} \tag{5.1}$$

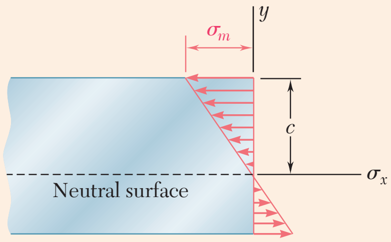


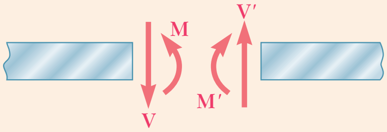
Fig. 5.21 Linear normal stress distribution for bending.

where I is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the bending couple **M** and c is the maximum distance from the neutral surface (Fig. 5.21). Introducing the elastic section modulus $S = I/c$ of the beam, the maximum value σ_m of the normal stress in the section can be expressed also as

$$\sigma_m = \frac{|M|}{S} \tag{5.3}$$

Shear and Bending-Moment Diagrams

From Eq. (5.1) it is seen that the maximum normal stress occurs in the section where $|M|$ is largest and at the point farthest from the neutral



(*a*) Internal forces
(positive shear and positive bending moment)

Fig. 5.22 Positive sign convention for internal shear and bending moment.

axis. The determination of the maximum value of $|M|$ and of the critical section of the beam in which it occurs is simplified if *shear diagrams* and *bending-moment diagrams are drawn*. These diagrams represent the variation of the shear and of the bending moment along the beam and are obtained by determining the values of V and M at selected points of the beam. These values are found by passing a section through the point and drawing the free-body diagram of either of the portions of beam. To avoid any confusion regarding the sense of the shearing force **V** and of the bending couple **M** (which act in opposite sense on the two portions of the beam), we follow the sign convention adopted earlier, as illustrated in Fig. 5.22.

Relationships Between Load, Shear, and Bending Moment

The construction of the shear and bending-moment diagrams is facilitated if the following relations are taken into account. Denoting by w the distributed load per unit length (assumed positive if directed downward)

$$\frac{dV}{dx} = -w \tag{5.5}$$

$$\frac{dM}{dx} = V \tag{5.7}$$

or in integrated form,

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \tag{5.6b}$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \tag{5.8b}$$

Equation (5.6b) makes it possible to draw the shear diagram of a beam from the curve representing the distributed load on that beam and V at one end of the beam. Similarly, Eq. (5.8b) makes it possible to draw the bending-moment diagram from the shear diagram and M at one end of the beam. However, concentrated loads introduce discontinuities in the shear diagram and concentrated couples in the bending-moment diagram, none of which is accounted for in these equations. The points of the beam where the bending moment is maximum or minimum are also the points where the shear is zero (Eq. 5.7).

Design of Prismatic Beams

Having determined σ_{all} for the material used and assuming that the design of the beam is controlled by the maximum normal stress in the beam, the minimum allowable value of the section modulus is

$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} \tag{5.9}$$

For a timber beam of rectangular cross section, $S = \frac{1}{6}bh^2$, where b is the width of the beam and h its depth. The dimensions of the section, therefore, must be selected so that $\frac{1}{6}bh^2 \geq S_{min}$.

For a rolled-steel beam, consult the appropriate table in Appendix C. Of the available beam sections, consider only those with a section modulus $S \geq S_{min}$. From this group we normally select the section with the smallest weight per unit length.

Singularity Functions

An alternative method to determine the maximum values of the shear and bending moment is based on the *singularity functions* $\langle x - a\rangle^n$. For $n \geq 0$,

$$\langle x - a\rangle^n = \begin{cases} (x-a)^n & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases} \tag{5.14}$$

Step Function

Whenever the quantity between brackets is positive or zero, the brackets should be replaced by ordinary parentheses, and whenever that quantity is negative, the bracket itself is equal to zero. Also, singularity functions can be integrated and differentiated as ordinary binomials. The singularity function corresponding to $n = 0$ is discontinuous at $x = a$ (Fig. 5.23). This function is called the *step function*.

$$\langle x - a\rangle^0 = \begin{cases} 1 & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases} \tag{5.15}$$

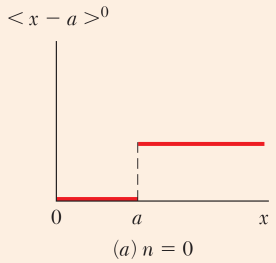


Fig. 5.23 Singular step function.

Using Singularity Functions to Express Shear and Bending Moment

The use of singularity functions makes it possible to represent the shear or the bending moment in a beam by a single expression. This is valid at any point of the beam. For example, the contribution to the shear of the concentrated load **P** applied at the midpoint C of a simply supported beam (Fig. 5.24) can be represented by $-P\langle x - \frac{1}{2}L\rangle^0$, since this expression is equal to zero to the left of C and to $-P$ to the right of C. Adding the reaction $R_A = \frac{1}{2}P$ at A, the shear at any point is

$$V(x) = \tfrac{1}{2}P - P\langle x - \tfrac{1}{2}L\rangle^0$$

The bending moment, obtained by integrating, is

$$M(x) = \tfrac{1}{2}Px - P\langle x - \tfrac{1}{2}L\rangle^1$$

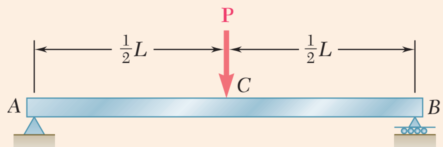


Fig. 5.24 Simply supported beam with a concentrated load at midpoint *C*.

Equivalent Open-Ended Loadings

The singularity functions representing the load, shear, and bending moment corresponding to various basic loadings were given in Fig. 5.16. A distributed load that does not extend to the right end of the beam or is discontinuous should be replaced by an equivalent combination of open-ended loadings. For instance, a uniformly distributed load extending from $x = a$ to $x = b$ (Fig. 5.25) is

$$w(x) = w_0\langle x - a\rangle^0 - w_0\langle x - b\rangle^0$$

The contribution of this load to the shear and bending moment is obtained through two successive integrations. Care should be used to include for $V(x)$ the contribution of concentrated loads and reactions, and for $M(x)$ the contribution of concentrated couples.

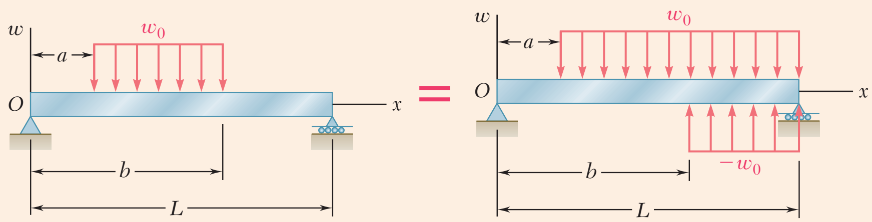


Fig. 5.25 Use of open-ended loadings to create a closed-ended loading.

Nonprismatic Beams

Nonprismatic beams are beams of variable cross section. By selecting the shape and size of the cross section so that its elastic section modulus $S = I/c$ varies along the beam in the same way as the bending moment M, beams can be designed where σ_m at each section is equal to σ_{all}. These are called *beams of constant strength,* and they provide a more effective use of the material than prismatic beams. Their section modulus at any section along the beam is

$$S = \frac{M}{\sigma_{all}} \tag{5.18}$$