Review and Summary

This chapter was devoted to the analysis and design of *shafts* subjected to twisting couples, or *torques*. Except for the last two sections of the chapter, our discussion was limited to *circular shafts*.

Deformations in Circular Shafts

The distribution of stresses in the cross section of a circular shaft is *statically indeterminate*. The determination of these stresses requires a prior analysis of the *deformations* occurring in the shaft [Sec. 3.1B]. In a circular shaft subjected to torsion, *every cross section remains plane and undistorted*. The *shearing strain* in a small element with sides parallel and perpendicular to the axis of the shaft and at a distance ρ from that axis is

$$\gamma = \frac{\rho\phi}{L} \qquad \textbf{(3.2)}$$

where ϕ is the angle of twist for a length L of the shaft (Fig. 3.55). Equation (3.2) shows that the *shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft*. It follows that the strain is maximum at the surface of the shaft, where ρ is equal to the radius c of the shaft:

$$\gamma_{\max} = \frac{c\phi}{L} \qquad \gamma = \frac{\rho}{c}\gamma_{\max} \qquad \textbf{(3.3, 4)}$$

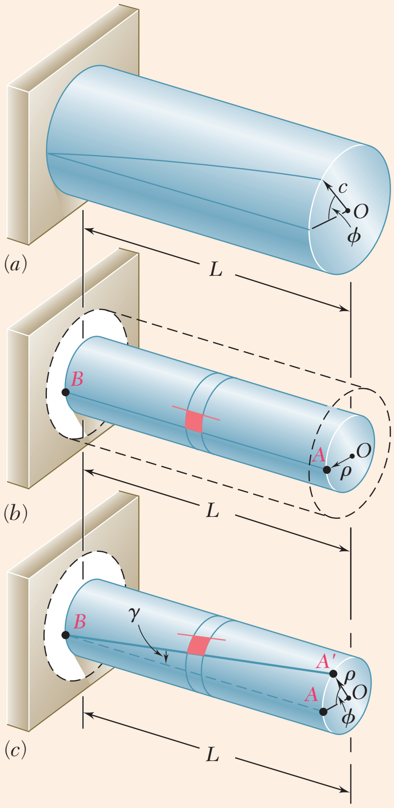

Fig. 3.55 Torsional deformations. (*a*) The angle of twist ϕ. (*b*) Undeformed portion of shaft of radius ρ. (*c*) Deformed portion of shaft; angle of twist ϕ and shearing strain γ share same arc length *AA′*.

Shearing Stresses in Elastic Range

The relationship between *shearing stresses* in a circular shaft within the elastic range [Sec. 3.1C] and Hooke's law for shearing stress and strain, $\tau = G\gamma$, is

$$\tau = \frac{\rho}{c}\tau_{\max} \qquad \textbf{(3.6)}$$

which shows that within the elastic range, the *shearing stress τ in a circular shaft also varies linearly with the distance from the axis of the shaft*. Equating the sum of the moments of the elementary forces exerted on any section of the shaft to the magnitude T of the torque applied to the shaft, the *elastic torsion formulas* are

$$\tau_{\max} = \frac{Tc}{J} \qquad \tau = \frac{T\rho}{J} \qquad \textbf{(3.9, 10)}$$

where c is the radius of the cross section and J its centroidal polar moment of inertia. $J = \frac{1}{2}\pi c^4$ for a solid shaft, and $J = \frac{1}{2}\pi(c_2^4 - c_1^4)$ for a hollow shaft of inner radius c_1 and outer radius c_2.

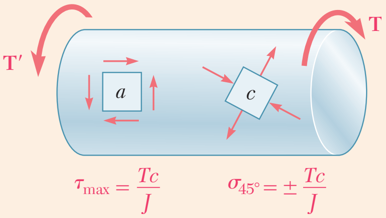

Fig. 3.56 Shaft elements with only shearing stresses or normal stresses.

We noted that while the element a in Fig. 3.56 is in pure shear, the element c in the same figure is subjected to normal stresses of the same magnitude,

Tc/J, with two of the normal stresses being tensile and two compressive. This explains why in a torsion test ductile materials, which generally fail in shear, will break along a plane perpendicular to the axis of the specimen, while brittle materials, which are weaker in tension than in shear, will break along surfaces forming a 45° angle with that axis.

Angle of Twist

Within the elastic range, the angle of twist ϕ of a circular shaft is proportional to the torque T applied to it (Fig. 3.57).

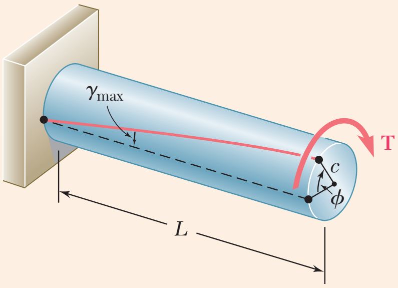

Fig. 3.57 Torque applied to fixed end shaft resulting in angle of twist ϕ.

$$\phi = \frac{TL}{JG} \quad \text{(units of radians)} \tag{3.15}$$

where L = length of shaft
J = polar moment of inertia of cross section
G = modulus of rigidity of material
ϕ is in *radians*

If the shaft is subjected to torques at locations other than its ends or consists of several parts of various cross sections and possibly of different materials, the angle of twist of the shaft must be expressed as the *algebraic sum* of the angles of twist of its component parts:

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i} \tag{3.16}$$

When both ends of a shaft BE rotate (Fig. 3.58), the angle of twist is equal to the *difference* between the angles of rotation ϕ_B and ϕ_E of its ends. When two shafts AD and BE are connected by gears A and B, the torques applied by gear A on shaft AD and gear B on shaft BE are *directly proportional* to the radii r_A and r_B of the two gears—since the forces applied on each other by the gear teeth at C are equal and opposite. On the other hand, the angles ϕ_A and ϕ_B are *inversely proportional* to r_A and r_B—since the arcs CC' and CC'' described by the gear teeth are equal.

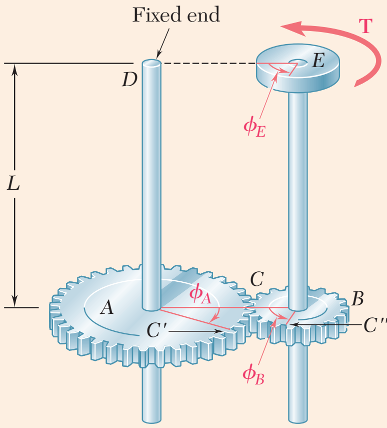

Fig. 3.58 Angles of twist at *E*, gear *B*, and gear *A* for a meshed-gear system.

Statically Indeterminate Shafts

If the reactions at the supports of a shaft or the internal torques cannot be determined from statics alone, the shaft is said to be *statically indeterminate*. The equilibrium equations obtained from free-body diagrams must be complemented by relationships involving deformations of the shaft and obtained from the geometry of the problem.

Transmission Shafts

For the *design of transmission shafts*, the power P transmitted is

$$P = 2\pi f T \tag{3.19}$$

where T is the torque exerted at each end of the shaft and f the *frequency* or speed of rotation of the shaft. The unit of frequency is the revolution per second (s^{-1}) or hertz (Hz). If SI units are used, T is expressed in newton-meters (N·m) and P in *watts* (W). If U.S. customary units are used, T is expressed in lb·ft or lb·in., and P in ft·lb/s or in·lb/s; the power can be converted into *horsepower* (hp) through

$$1 \text{ hp} = 550 \text{ ft·lb/s} = 6600 \text{ in·lb/s}$$

To design a shaft to transmit a given power P at a frequency f, solve Eq. (3.19) for T. This value and the maximum allowable value of τ for the material can be used with Eq. (3.9) to determine the required shaft diameter.

Stress Concentrations

Stress concentrations in circular shafts result from an abrupt change in the diameter of a shaft and can be reduced through the use of a *fillet* (Fig. 3.59). The maximum value of the shearing stress at the fillet is

$$\tau_{\max} = K\frac{Tc}{J} \tag{3.22}$$

where the stress Tc/J is computed for the smaller-diameter shaft and K is a stress concentration factor.

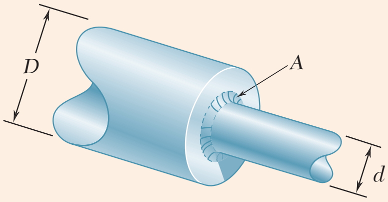

Fig. 3.59 Shafts having two different diameters with a fillet at the junction.

Plastic Deformations

Even when Hooke's law does not apply, the distribution of *strains* in a circular shaft is always linear. If the shearing-stress-strain diagram for the material is known, it is possible to plot the shearing stress τ against the distance ρ from the axis of the shaft for any given value of $\tau_{\max}$ (Fig. 3.60). Summing the torque of annular elements of radius ρ and thickness $d\rho$, the torque T is

$$T = \int_0^c \rho\tau(2\pi\rho\, d\rho) = 2\pi\int_0^c \rho^2\tau\, d\rho \tag{3.23}$$

where τ is the function of ρ plotted in Fig. 3.60.

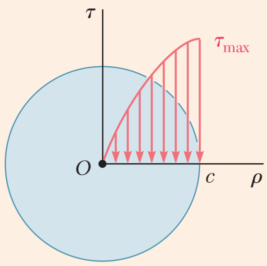

Fig. 3.60 Shearing stress distribution for shaft with nonlinear stress-strain response.

Modulus of Rupture

An important value of the torque is the ultimate torque T_U, which causes failure of the shaft. This can be determined either experimentally, or by Eq. (3.22) with $\tau_{\max}$ chosen equal to the ultimate shearing stress τ_U of the

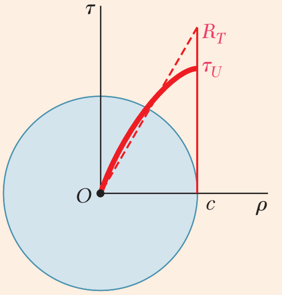

Fig. 3.61 Stress distribution in circular shaft at failure.

material. From T_U, and assuming a linear stress distribution (Fig 3.61), we determined the corresponding fictitious stress $R_T = T_U c/J$, known as the *modulus of rupture in torsion*.

Solid Shaft of Elastoplastic Material

In a *solid circular shaft* made of an *elastoplastic material*, as long as τ_{max} does not exceed the yield strength τ_Y of the material, the stress distribution across a section of the shaft is linear (Fig. 3.62*a*). The torque T_Y corresponding to $\tau_{max} = \tau_Y$ (Fig. 3.62*b*) is the *maximum elastic torque*. For a solid circular shaft of radius *c*,

$$T_Y = \tfrac{1}{2}\pi c^3 \tau_Y \tag{3.26}$$

As the torque increases, a plastic region develops in the shaft around an elastic core of radius ρ_Y. The torque T corresponding to a given value of ρ_Y is

$$T = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\rho_Y^3}{c^3}\right) \tag{3.29}$$

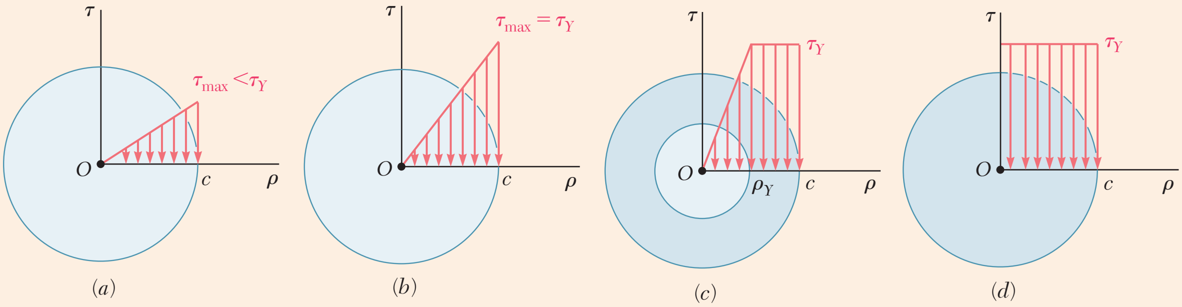

Fig. 3.62 Stress distributions for elastoplastic shaft at different stages of loading: (*a*) elastic, (*b*) impending yield, (*c*) partially yielded, and (*d*) fully yielded.

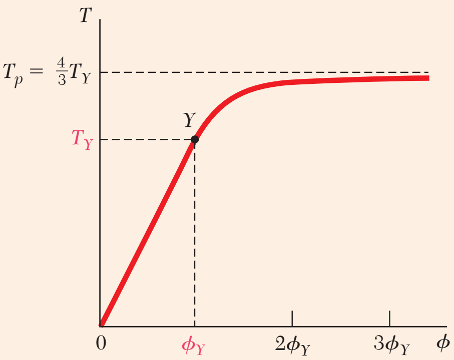

Fig. 3.63 Load-displacement relation for elastoplastic material.

As ρ_Y approaches zero, the torque approaches a limiting value T_p, called the *plastic torque*:

$$T_p = \frac{4}{3}T_Y \tag{3.30}$$

Plotting the torque T against the angle of twist ϕ of a solid circular shaft (Fig. 3.63), the segment of straight line $0Y$ defined by Eq. (3.15) and followed by a curve approaching the straight line $T = T_p$ is

$$T = \frac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\phi_Y^3}{\phi^3}\right) \tag{3.34}$$

Permanent Deformation and Residual Stresses

Loading a circular shaft beyond the onset of yield and unloading it results in a *permanent deformation* characterized by the angle of twist $\phi_p = \phi - \phi'$, where ϕ corresponds to the loading phase described in the previous paragraph, and ϕ' to the unloading phase represented by a straight line in

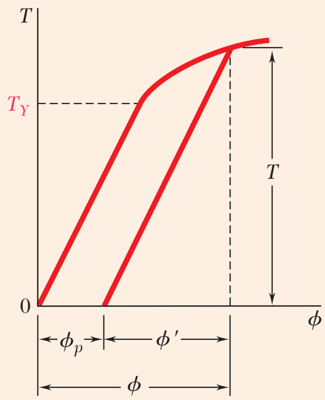

Fig. 3.64 Torque-angle of twist response for loading past yield and, followed by unloading.

Fig. 3.64. *Residual stresses* in the shaft can be determined by adding the maximum stresses reached during the loading phase and the reverse stresses corresponding to the unloading phase.

Torsion of Noncircular Members

The equations for the distribution of strain and stress in circular shafts are based on the fact that due to the axisymmetry of these members, cross sections remain plane and undistorted. This property does not hold for noncircular members, such as the square bar of Fig. 3.65.

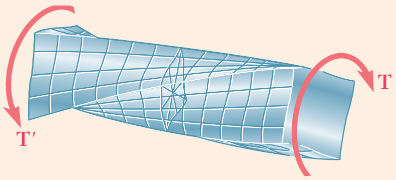

Fig. 3.65 Twisting a shaft of square cross section.

Bars of Rectangular Cross Section

For straight bars with a *uniform rectangular cross section* (Fig. 3.66), the maximum shearing stress occurs along the center line of the *wider* face of the bar. The *membrane analogy* can be used to visualize the distribution of stresses in a noncircular member.

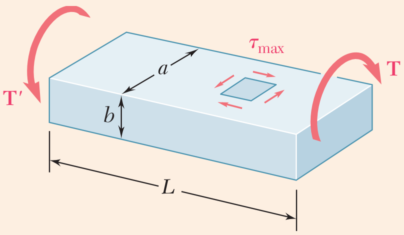

Fig. 3.66 Shaft with rectangular cross section, showing the location of maximum shearing stress.

Thin-Walled Hollow Shafts

The shearing stress in *noncircular thin-walled hollow shafts* is parallel to the wall surface and varies both across and along the wall cross section. Denoting the average value of the shearing stress τ, computed across the wall at a given point of the cross section, and by t the thickness of the wall at that point (Fig. 3.67), we demonstrated that the product $q = \tau t$, called the *shear flow*, is constant along the cross section.

The average shearing stress τ at any given point of the cross section is

$$\tau = \frac{T}{2t\mathcal{a}} \tag{3.50}$$

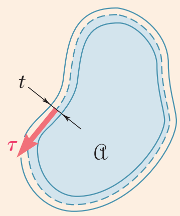

Fig. 3.67 Area for shear flow.