Review and Summary

Normal Strain

Consider a rod of length L and uniform cross section, and its deformation δ under an axial load **P** (Fig. 2.59). The *normal strain* ϵ in the rod is defined as the *deformation per unit length*:

$$\epsilon = \frac{\delta}{L} \qquad \textbf{(2.1)}$$

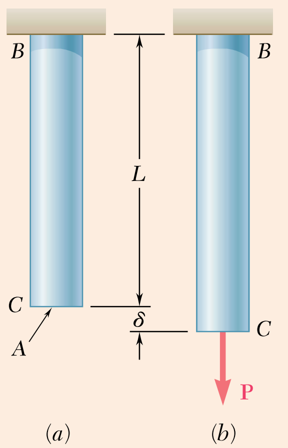

Fig. 2.59 Undeformed and deformed axially-loaded rod.

In the case of a rod of variable cross section, the normal strain at any given point Q is found by considering a small element of rod at Q:

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \qquad \textbf{(2.2)}$$

Stress-Strain Diagram

A *stress-strain diagram* is obtained by plotting the stress σ versus the strain ϵ as the load increases. These diagrams can be used to distinguish between *brittle* and *ductile* materials. A brittle material ruptures without any noticeable prior change in the rate of elongation (Fig. 2.60), while a ductile material

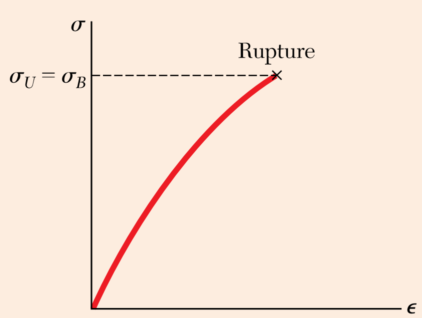

Fig. 2.60 Stress-strain diagram for a typical brittle material.

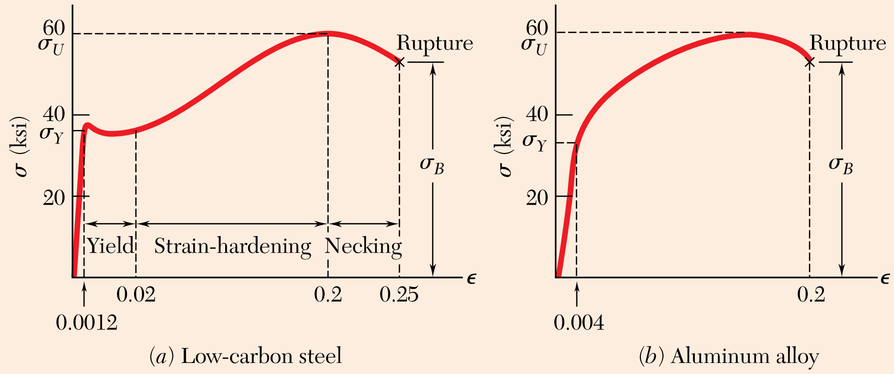

Fig. 2.61 Stress-strain diagrams of two typical ductile metal materials.

yields after a critical stress σ_Y (the *yield strength*) has been reached (Fig. 2.61). The specimen undergoes a large deformation before rupturing, with a relatively small increase in the applied load. An example of brittle material with different properties in tension and compression is *concrete*.

Hooke's Law and Modulus of Elasticity

The initial portion of the stress-strain diagram is a straight line. Thus, for small deformations, the stress is directly proportional to the strain:

$$\sigma = E\epsilon \tag{2.6}$$

This relationship is *Hooke's law*, and the coefficient E is the *modulus of elasticity* of the material. The *proportional limit* is the largest stress for which Eq. (2.4) applies.

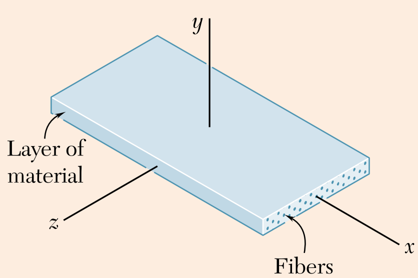

Fig. 2.62 Layer of fiber-reinforced composite material.

Properties of *isotropic* materials are independent of direction, while properties of *anisotropic* materials depend upon direction. *Fiber-reinforced composite materials* are made of fibers of a strong, stiff material embedded in layers of a weaker, softer material (Fig. 2.62).

Elastic Limit and Plastic Deformation

If the strains caused in a test specimen by the application of a given load disappear when the load is removed, the material is said to behave *elastically*. The largest stress for which this occurs is called the *elastic limit* of the material. If the elastic limit is exceeded, the stress and strain decrease in a linear fashion when the load is removed, and the strain does not return to zero (Fig. 2.63), indicating that a *permanent set* or *plastic deformation* of the material has taken place.

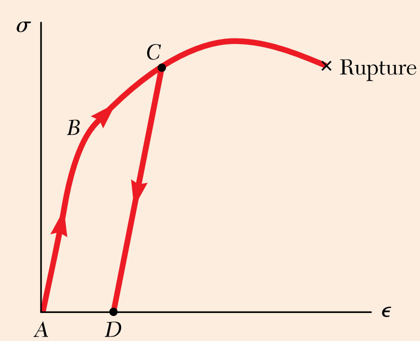

Fig. 2.63 Stress-strain response of ductile material loaded beyond yield and unloaded.

Fatigue and Endurance Limit

Fatigue causes the failure of structural or machine components after a very large number of repeated loadings, even though the stresses remain in the elastic range. A standard fatigue test determines the number n of successive loading-and-unloading cycles required to cause the failure of a specimen for any given maximum stress level σ and plots the resulting σ-n curve. The value of σ for which failure does not occur, even for an indefinitely large number of cycles, is known as the *endurance limit*.

Elastic Deformation Under Axial Loading

If a rod of length L and uniform cross section of area A is subjected at its end to a centric axial load **P** (Fig. 2.64), the corresponding deformation is

$$\delta = \frac{PL}{AE} \tag{2.9}$$

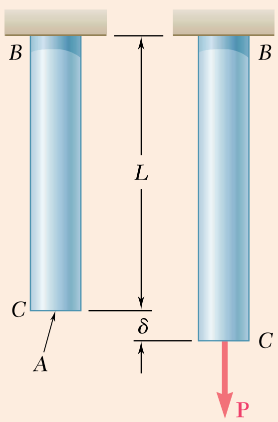

Fig. 2.64 Undeformed and deformed axially-loaded rod.

If the rod is loaded at several points or consists of several parts of various cross sections and possibly of different materials, the deformation δ of the rod must be expressed as the sum of the deformations of its component parts:

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \tag{2.10}$$

Statically Indeterminate Problems

Statically indeterminate problems are those in which the reactions and the internal forces *cannot* be determined from statics alone. The equilibrium equations derived from the free-body diagram of the member under consideration were complemented by relations involving deformations and obtained from the geometry of the problem. The forces in the rod and in the tube of Fig. 2.65, for instance, were determined by observing that their sum is equal to P, and that they cause equal deformations in the rod and in the tube. Similarly, the reactions at the supports of the bar of

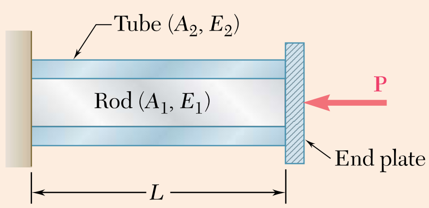

Fig. 2.65 Statically indeterminate problem where concentric rod and tube have same strain but different stresses.

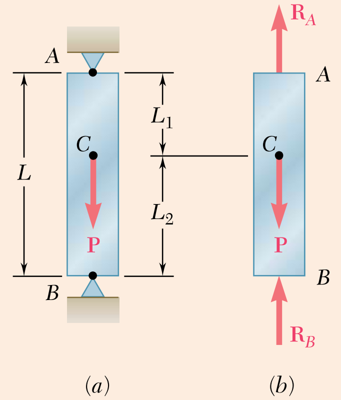

Fig. 2.66 (*a*) Axially-loaded statically-indeterminate member. (*b*) Free-body diagram.

Fig. 2.66 could not be obtained from the free-body diagram of the bar alone, but they could be determined by expressing that the total elongation of the bar must be equal to zero.

Problems with Temperature Changes

When the temperature of an *unrestrained rod AB* of length L is increased by ΔT, its elongation is

$$\delta_T = \alpha(\Delta T)L \quad \textbf{(2.13)}$$

where α is the *coefficient of thermal expansion* of the material. The corresponding strain, called *thermal strain*, is

$$\epsilon_T = \alpha \Delta T \quad \textbf{(2.14)}$$

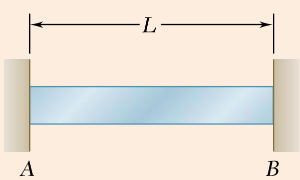

Fig. 2.67 Fully restrained bar of length *L*.

and *no stress* is associated with this strain. However, if rod *AB* is *restrained* by fixed supports (Fig. 2.67), stresses develop in the rod as the temperature increases, because of the reactions at the supports. To determine the magnitude P of the reactions, the rod is first detached from its support at B (Fig. 2.68*a*).

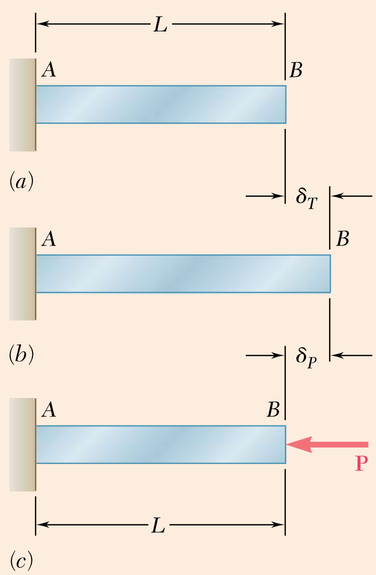

Fig. 2.68 Determination of reactions for bar of Fig. 2.67 subject to a temperature increase. (*a*) Support at *B* removed. (*b*) Thermal expansion. (*c*) Application of support reaction to counter thermal expansion.

The deformation δ_T of the rod occurs as it expands due to of the temperature change (Fig. 2.68*b*). The deformation δ_P caused by the force **P** is required to bring it back to its original length, so that it may be reattached to the support at *B* (Fig. 2.68*c*).

Lateral Strain and Poisson's Ratio

When an axial load **P** is applied to a homogeneous, slender bar (Fig. 2.69), it causes a strain, not only along the axis of the bar but in any transverse direction. This strain is the *lateral strain*, and the ratio of the lateral strain over the axial strain is called *Poisson's ratio*:

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \qquad \textbf{(2.17)}$$

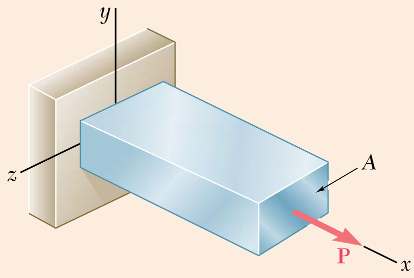

Fig. 2.69 A bar in uniaxial tension.

Multiaxial Loading

The condition of strain under an axial loading in the *x* direction is

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \qquad \textbf{(2.19)}$$

A *multiaxial loading* causes the state of stress shown in Fig. 2.70. The resulting strain condition was described by the *generalized Hooke's law* for a multiaxial loading.

$$\begin{aligned} \epsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \epsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ \epsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \end{aligned} \qquad \textbf{(2.20)}$$

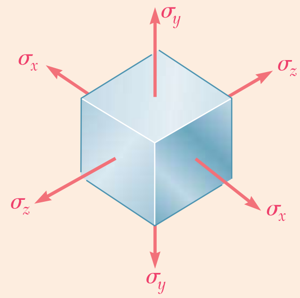

Fig. 2.70 State of stress for multiaxial loading.

Dilatation

If an element of material is subjected to the stresses σ_x, σ_y, σ_z, it will deform and a certain change of volume will result. The *change in volume per unit volume* is the *dilatation* of the material:

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \qquad \textbf{(2.22)}$$

Bulk Modulus

When a material is subjected to a hydrostatic pressure *p*,

$$e = -\frac{p}{k} \qquad \textbf{(2.25)}$$

where *k* is the *bulk modulus* of the material:

$$k = \frac{E}{3(1 - 2\nu)} \qquad \textbf{(2.24)}$$

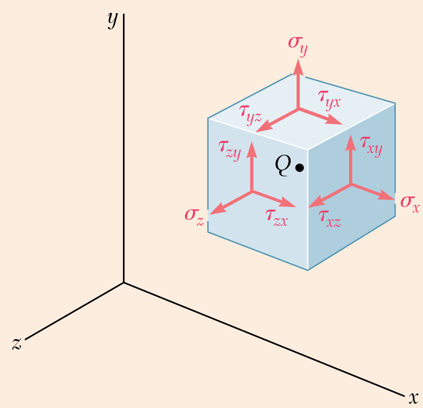

Fig. 2.71 Positive stress components at point Q for a general state of stress.

Shearing Strain: Modulus of Rigidity

The state of stress in a material under the most general loading condition involves shearing stresses, as well as normal stresses (Fig. 2.71). The shearing stresses tend to deform a cubic element of material into an oblique parallelepiped. The stresses τ_{xy} and τ_{yx} shown in Fig. 2.72 cause the angles formed by the faces on which they act to either increase or decrease by a small angle γ_{xy}. This angle defines the *shearing strain* corresponding to the x and y directions. Defining in a similar way the shearing strains γ_{yz} and γ_{zx}, the following relations were written:

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \qquad \textbf{(2.27, 28)}$$

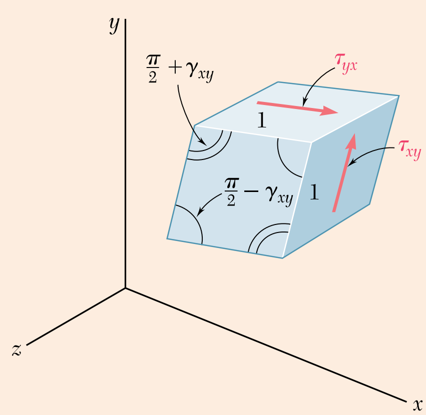

Fig. 2.72 Deformation of unit cubic element due to shearing stress.

which are valid for any homogeneous isotropic material within its proportional limit in shear. The constant G is the *modulus of rigidity* of the material, and the relationships obtained express *Hooke's law for shearing stress and strain*. Together with Eqs. (2.20), they form a group of equations representing the generalized Hooke's law for a homogeneous isotropic material under the most general stress condition.

While an axial load exerted on a slender bar produces only normal strains—both axial and transverse—on an element of material oriented along the axis of the bar, it will produce both normal and shearing strains on an element rotated through 45° (Fig. 2.73). The three constants E, ν, and G are not independent. They satisfy the relation

$$\frac{E}{2G} = 1 + \nu \tag{2.34}$$

This equation can be used to determine any of the three constants in terms of the other two.

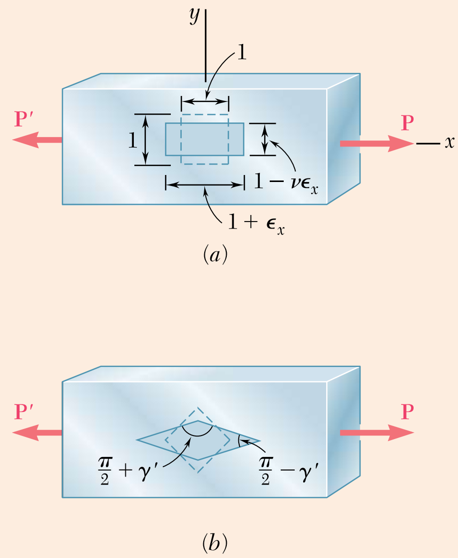

Fig. 2.73 Representations of strain in an axially-loaded bar: (*a*) cubic strain element with faces aligned with coordinate axes; (*b*) cubic strain element with faces rotated 45° about *z*-axis.

Saint-Venant's Principle

Saint-Venant's principle states that except in the immediate vicinity of the points of application of the loads, the distribution of stresses in a given member is independent of the actual mode of application of the loads. This principle makes it possible to assume a uniform distribution of stresses in a member subjected to concentrated axial loads, except close to the points of application of the loads, where stress concentrations will occur.

Stress Concentrations

Stress concentrations will also occur in structural members near a discontinuity, such as a hole or a sudden change in cross section. The ratio of the maximum value of the stress occurring near the discontinuity over the average stress computed in the critical section is referred to as the *stress-concentration factor* of the discontinuity:

$$K = \frac{\sigma_{\text{max}}}{\sigma_{\text{ave}}} \tag{2.40}$$

Plastic Deformations

Plastic deformations occur in structural members made of a ductile material when the stresses in some part of the member exceed the yield strength of the material. An idealized *elastoplastic material* is characterized by the stress-strain diagram shown in Fig. 2.74. When an indeterminate structure

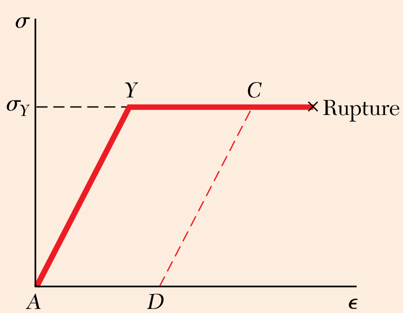

Fig. 2.74 Stress-strain diagram for an idealized elastoplastic material.

undergoes plastic deformations, the stresses do not, in general, return to zero after the load has been removed. The stresses remaining in the various parts of the structure are called *residual stresses* and can be determined by adding the maximum stresses reached during the loading phase and the reverse stresses corresponding to the unloading phase.